Kinematic factor derivation

Presented here is a full derivation of the kinematic factor using in ion scattering. Figure 1 shows a schematic of the binary collision model used in ion scattering.

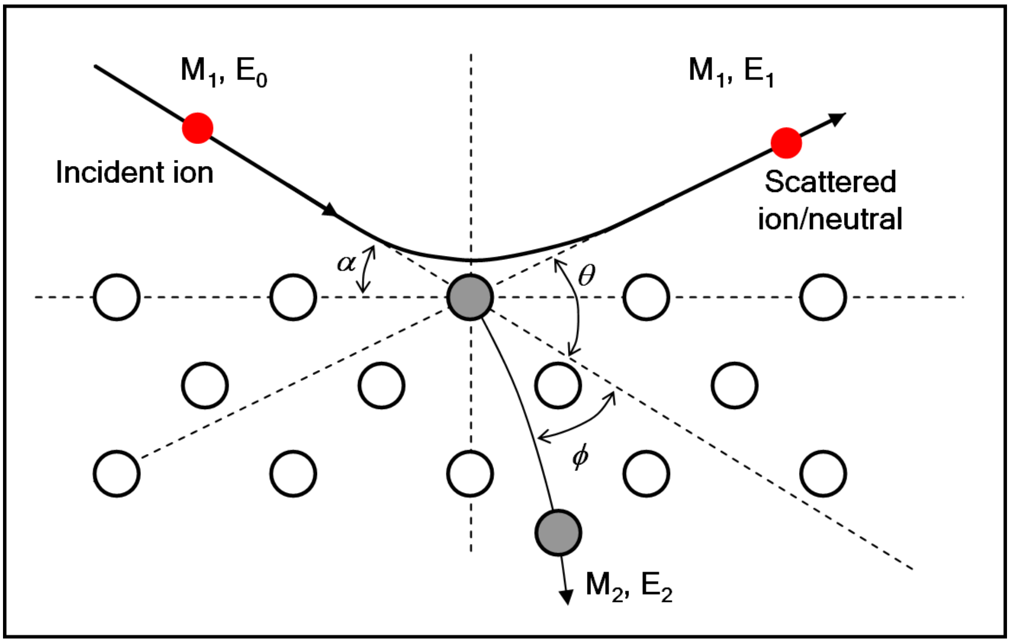

Figure 1: Schematic demonstration of an ion scattering from a surface atom. An ion with mass m_1 and kinetic energy E_0 (velocity v_0) is incident at an angle α on a target atom with mass m_2. The ion is scattered through an angle θ with respect to the incident direction, losing kinetic energy to the target atom during the collision. The scattered ion possesses a kinetic energy E_1 (velocity v_1), whilst the target atom recoils with a kinetic energy E_2 (velocity v_2) at an angle ϕ with respect to the incident direction.

Conservation of energy gives:

$$m_1 v_0^2 = m_1 v_1^2 + m2 v_2^2 \tag{1}$$

Rearranging this equation leads to:

$$\boxed{v_2^2 = m_1(v_0^2 - v_1^2)} \tag{2}$$

Conservation of momentum parallel to the incident direction can be expressed by:

$$m_1 v_0 = m_1 v_1 \cos\theta + m_2 v_2 \cos\phi \tag{3}$$

This can be rearranged and squared to give:

$$m_1^2 v_0^2 + m_1^2 v_1^2 \cos^2\theta - 2m_1^2 v_0 v_1 \cos\theta = m_2^2 v_2^2 \cos^2\phi \tag{4}$$

Momentum perpendicular to the incident direction is given by:

$$0 = m_1 v_1 \sin\theta - m_2 v_2 \sin\phi \tag{5}$$

Rearranging this equation and squaring gives:

$$m_1^2 v_1^2 \sin^2\theta = m_2^2 v_2^2 \sin^2\phi \tag{6}$$

Adding equations 4 and 6 gives:

$$m_1^2 v_0^2 + m_1^2 v_1^2 (\cos^2\theta + \sin^2\theta) - 2m_1^2 v_0 v_1 \cos\theta = m_2^2 v_2^2 (\cos^2\phi + \sin^2\phi) \tag{7}$$

Recognising that $\cos^2\theta + \sin^2\theta = 1$ leads to:

$$m_1^2 v_0^2 + m_1^2 v_1^2 - 2m_1^2 v_0 v_1 \cos\theta = m_2^2 v_2^2 \tag{8}$$

Substituting equation 2 into the right hand side of equation 8 gives:

$$m_1^2 v_0^2 + m_1^2 v_1^2 - 2m_1^2 v_0 v_1 \cos\theta = m_2 m_1 (v_0^2 - v_1^2) \tag{9}$$

Dividing both sides by $m_1^2 v_0^2$ gives:

$$1 + \frac{v_1^2}{v_0^2} - \frac{2v_1 \cos\theta}{v_0} = \frac{m_2}{m_1}(1 - \frac{v_1^2}{v_0^2}) \tag{10}$$

Setting $m_2/m_1 =$ A and rearranging gives:

$$\frac{v_1^2}{v_0^2}(1 + A) - \frac{2v_1 \cos\theta}{v_0} + (1 - A) = 0 \tag{11}$$

This is a quadratic in v_1/v_0. Using the quadratic formula:

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{12}$$

with X = v_1/v_0, a = (1 + A), b = 2$\cos\theta$ and c = (1 - A) leads to:

$$\frac{v_1}{v_0} = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4(1+A)(1-A)}}{2(1+A)} \tag{13}$$

This reduces to:

$$\frac{v_1}{v_0} = \frac{\cos\theta \pm \sqrt{\cos^2\theta - (1+A)(1-A)}}{(1+A)} \tag{14}$$

Multiplying out the terms included in the square root gives:

$$\frac{v_1}{v_0} = \frac{\cos\theta \pm \sqrt{A^2 + \cos^2\theta - 1}}{(1+A)} \tag{15}$$

However, $cos^2\theta$ - 1 = - $\sin^2\theta$, so:

$$\frac{v_1}{v_0} = \frac{\cos\theta \pm \sqrt{A^2 - \sin^2\theta}}{(1+A)} \tag{16}$$

As the kinematic factor, k = E_1 / E_0 = $m_1 v_1^2$ / $m_1 v_0^2$, then:

$$\boxed{k = \frac{E_1}{E_0} = \frac{1}{(1+A)^2}(\cos\theta \pm \sqrt{A^2 - \sin^2\theta})^2} \tag{17}$$

For CAICISS, θ = 180°, so:

$$\boxed{k = \frac{(A-1)^2}{(A+1)^2}} \tag{18}$$